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NONEXISTENCE OF UNIVALUED INTEGRALS AND BRANCHING OF SOLUTIONS IN SOLID BODY DYNAMICS

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Investigations of Kowalewska, Liapunov and other authors in the field of solid body dynamics have shown that a general solution of equations of motion can be represented by univalued functions of time only in the classical problems of Euler, Lagrange, and Kowalewska, and then only when there exists a supplementary univalued integral. Whether this is purely coincidental or due to some fundamental causes remains obscure. It is shown below by using Poincare's method of the small parameter that it is the existence of an infinite number of non-unique solutions which generally hinders the appearance of a univalued analytic integral.

1. Theorem on the nonexistence of univalued integrals. Let us consider the canonical system of differential equations with the Hamiltonian

$$H(I, \varphi, \mu) = H_0(I) + \mu H_1(I, \varphi) + \ldots \qquad (1.1)$$
$$I = (I_1, I_2), \quad \varphi = (\varphi_1, \varphi_2)$$

where $H(I, \varphi, \mu)$ is assumed to be a real analytic function in the direct product $D \times T^2\{\varphi \bmod 2\pi\} \times (-\varepsilon, \varepsilon)$ (D is a region in $R^2\{I_1, I_2\}$).

We assume that for fixed $I \in D$, and $\mu \in (-\varepsilon, \varepsilon)$ the Hamiltonian (1.1) is continued in the direct product of complex planes $\mathbf{C} \times \mathbf{C}$ to a univalued function of variables φ_1 and φ_2. Presence of singular points of function (1.1) is not excluded when φ_1 and φ_2 are complex.

Let us define some of the notation. Let V be a compact subregion of D and $\nu > 0$. Then $\Delta(V, \nu) = \{I : I = I' + iI'', I' \in V, |I''| < \nu\}$. If $V' \subset V$ and $\nu' < \nu$, then $\Delta(V', \nu') \subset \Delta(V, \nu)$. We set $\Pi(\rho) = \{(\varphi_1, \varphi_2) \in \mathbf{C} \times \mathbf{C}: |\operatorname{Im} \varphi_k| < \rho;\ k = 1,2\}$. All solutions of the unperturbed system

$$I = I^\circ, \ \varphi = \varphi^\circ + \omega t \ (\omega = (\omega_1, \omega_2), \ \omega_k(I) = \partial H_0 / \partial I_k; \ k = 1,2)$$

are univalued functions of the complex variable $t \in \mathbf{C}$, but solutions of perturbed equations are generally not univalued when $\mu \neq 0$.

Let us consider in the complex time plane $t \in \mathbf{C}$ the closed continuous contour Γ and its image γ in the mapping $t \to \mathbf{C} \times \mathbf{C}\ (t \in \Gamma)$ according to formula

$$\varphi(t) = \varphi^\circ + \omega(I^\circ)t \quad (I^\circ \in D, \ \varphi^\circ \in T^2)$$

Let us assume that the Hamiltonian $H(I, \varphi, \mu)$ is analytic in the direct product $V \times \Omega \times (-\varepsilon, \varepsilon)$, where V is some compact neighborhood of point $I^\circ \in D$ and Ω

is a connected region in $\mathbf{C} \times \mathbf{C}$, $\Pi(s) \subset \Omega \subset \Pi(S)$ $(0 < s < S)$, that contains the continuous curve γ.

If $\varphi \in \Omega$, then

$$H(I, \varphi + 2\pi, \mu) = H(I, \varphi, \mu)$$

This equality is evidently valid for real values of φ. In the general case of $\varphi \in \Omega$ its validity follows from the connectedness of Ω and the uniqueness of the analytic continuation.

Note that when $\varphi \in \Omega$ and $\mu \in (-\varepsilon, \varepsilon)$ then function $H(I, \varphi, \mu)$ is analytic in region $\Delta(V, \nu)$ with respect to I_1 and I_2 if ν is fairly small.

According to Poincare's theorem [1, 2] solutions of perturbed equations can be expanded in power series of μ

$$\begin{aligned} I &= I^\circ + \mu I^1(t; I^\circ, \varphi^\circ) + \dots \\ \varphi &= \varphi^\circ + \omega(I^\circ)t + \mu\varphi^1(t; I^\circ, \varphi^\circ) + \dots \end{aligned} \tag{1.2}$$

which are convergent when $t \in \Gamma$ and parameter μ is fairly small.

We say that the analytic vector function $f(t)$, $t \in \Gamma$ is not univalued along Γ, if it has a discontinuity $\xi \neq 0$ after going around contour Γ. If function $I^1(t; I^\circ, \varphi^\circ)$ is not univalued along Γ, the perturbed solution (1.2) is also not univalued along contour Γ for small values of parameter μ.

Let us fix the initial values I° and φ° and continuously distort contour Γ so that contour γ does not pass through any singular points of the Hamiltonian $H(I, \varphi, \mu)$. Using the Cauchy theorem it is possible to show that function $I^1(t; I^\circ, \varphi^\circ)$ when passing around the distorted contour changes by the same quantity $\xi = (\xi_1, \xi_2) \neq 0$. Since according to initial data solutions (1.2) are continuous [1, 2], function $I^1(t; I^\circ, \varphi^\circ)$ is not univalued along the contour for all $I = I^\circ$ from some small region $U \subset D$, and when $I^\circ \in U$ the jump $\xi = \xi(I^\circ) \neq 0$ occurs.

We say that a system of canonical equations with Hamiltonian (1.1) has the univalued integral $F(I, \varphi, \mu)$, if that function

1) is the first integral;

2) is a real analytic function in region $D \times T^2 \times (-\varepsilon, \varepsilon)$, and

3) is univalued with respect to variables φ_1 and φ_2 in the direct product $\mathbf{C} \times \mathbf{C}$ for fixed I and μ.

Function H is evidently one of such integrals. We stress that the presence of singular points of function F when variables φ_1 and φ_2 are complex.

T h e o r e m 1. Let us assume that the unperturbed system is nondegenerate, i.e. that in region D, the Hessian

$$|\partial^2 H_0 / \partial I^2| \neq 0$$

and that for some $I^\circ \in D$ and $\varphi^\circ \in T^2$ functional $I^1(t; I^\circ, \varphi^\circ)$ is then not univalued along contour Γ. The canonical system of differential equations with

Hamiltonian (1.1) does not contain the univalued integral $F(I, \varphi, \mu)$, that is independent of (1.1) and analytic in the direct product $W \times \Omega \times (-\varepsilon, \varepsilon)$, where W is some neighborhood of point $I = I^\circ$.

2. **Proof of theorem** 1 (see [1] Chapter V). We expand function $F(I, \varphi, \mu)$ in series in powers of μ

$$F = F_0(I, \varphi) + \mu F_1(I, \varphi) + \ldots \tag{2.1}$$

If $(I, \varphi) \in \Delta(V, \nu) \times \Omega$ ($V \subset W$, and ν is fairly small), this series is convergent when parameter μ is small.

Lemma 1. Function F_0 is independent of φ.

This statement is implied by the nondegeneracy of the unperturbed system (see [1] Chapter V) when $(I, \varphi) \in D \times T^2$, by the connectedness of region Ω when $\varphi \in \Omega$.

Lemma 2. Functions $H_0(I)$ and $F_0(I)$ are interdependent in region W, i.e. the Jacobian

$$\frac{\partial(H_0, F_0)}{\partial(I_1, I_2)} \equiv 0 \tag{2.2}$$

when $I \in W$.

Proof. Since $F(I, \varphi, \mu)$ is the first integral of the canonical system of equations with Hamiltonian (1.1), that function is constant along solutions (1.2). Hence its values at the instant of time $\tau \in \Gamma$ and after passing over contou Γ are the same. From this

$$F_0(I^\circ + \mu I^1(\tau) + \ldots) + \mu F_1(I^\circ + \mu I^1(\tau) + \ldots, \varphi^\circ + \omega t + \mu\varphi^1(\tau) + \ldots) + \ldots \equiv F_0(I^\circ + \mu(I^1(\tau) + \xi(I^\circ)) + \ldots) + \mu F_1(I^\circ + \ldots, \varphi^\circ + \omega t + \ldots) + \ldots$$

Expanding this identity in power series in μ and equating the coefficient at the first power of μ, we obtain

$$\frac{\partial F_0}{\partial I_1}\xi_1 + \frac{\partial F_0}{\partial I_2}\xi_2 = 0$$

Since $H(I, \varphi, \mu)$ is also the first univalued integral, hence

$$\frac{\partial H_0}{\partial I_1}\xi_1 + \frac{\partial H_0}{\partial I_2}\xi_2 = 0$$

Comparing the last two formulas, we conclude that identity (2.2) is valid when $I^\circ \in W \cap U$, i.e. functions H_0 and F_0 are interdependent in region $W \cap U$ and, consequently, throughout region W.

Let us now assume that functions (1.1) and (2.1) are independent. Let J be the nonzero minor of the second order of Jacobi's matrix

$$\frac{\partial(H, F)}{\partial(I_1, I_2, \varphi_1, \varphi_2)}$$

Function $J(I, \varphi, \mu)$ is analytic and can be expanded in a convergent series in powers of μ. We assume that in that expansion the coefficient at μ^p ($p \geqslant 0$) is nonzero. It follows from Lemma 2 that $p \geqslant 1$.

Since the Hessian $|\partial^2 H_0/\partial I^2| \not\equiv 0$, hence in some small region $V \subset W \subset D$ the derivative $\partial H_0 / \partial I_1 \neq 0$. This shows that in this region equation $H_0(I_1, I_2) = H_0$ can be solved for I_1 and that the obtained expression can be substituted into function $F_0(I_1, I_2)$, yielding $F_0 = F_0(I_1^*(H_0, I_2), I_2)$. Since H_0 and F_0 are dependent, hence $F_0 = \Psi(H_0)$, where $\Psi(x)$ is an analytic function in the interval $(\min_V H_0, \max_V H_0)$. Note that $\Psi(z)$ is analytic in the small complex neighborhood of that integral.

When μ is fairly small function $\Psi(H)$ is analytic with respect to I and φ in region $V' \times \Omega$, where V' is a compact region inside V. Since the expansion of function $F - \Psi(H)$ in series in powers of μ does not contain a free term, hence $F - \Psi(H) = \mu F'$. Function $F'(I, \varphi, \mu)$ is the first univalued integral that is analytic in region $\Delta(V', \nu') \times \Omega \times (-\varepsilon', \varepsilon')$ where ν' and ε' are fairly small. By Lemmas 1 and 2 function F_0' does not contain φ, and $H_0(I)$ and $F_0'(I)$ are interdependent in region $V' \subset D$. Since $F = \Psi(H) + \mu F_0' + \mu^2 F_1' + \ldots$, hence the expansion of the minor J in series of powers of μ begins with a term of order μ^2.

Repeating this operation p times we find that the expansion of function J begins with terms of order μ^{p+1} and not μ^p as assumed above. This contradiction proves Theorem 1.

3. Application to the problem of heavy solid body rotation about a fixed point. The Hamiltonian of this problem can be represented in the form (see [1])

$$H = H_0 + \mu H_1 \tag{3.1}$$

where H_0 is the system kinetic energy (the Hamiltonian of the Euler — Poinsot problem) and μH_1 is the potential energy (μ is the product of the body weight by the distance between the suspension point and the center of gravity). We assume that parameter μ is small. In other words, the problem of rotation of a heavy solid body about a fixed point is considered as a perturbation of the integrable Euler — Poinsot problem.

Using the area integral we reduce the number of degrees of freedom to two, and assume henceforth that the constant of areas is fixed.

When $\mu = 0$ the unperturbed problem is integrable, and in terms of variables I and φ, action and angle, respectively, function H_0 depends only on $I = (I_1, I_2)$, as defined by the following implicit formula [3]:

$$I_1(H_0, I_2) = \frac{1}{2\pi} \oint \sqrt{\frac{2H_0 - I_2^2(A^{-1}\sin^2 x + B^{-1}\cos^2 x)}{C^{-1} - A^{-1}\sin^2 x - B^{-1}\cos^2 x}}\, dx$$

where $A \geqslant B \geqslant C$ are the principal moments of inertia of the solid body. Function $H_0(I_1, I_2)$ is determinate in region $\Delta = \{I_1, I_2 : I_2 \geqslant 0, |I_1| \leqslant I_2\}$, and ananlytic everywhere, except at points of the straight lines $I_1 = 0$, $|I_1| = I_2$, $2H_0 = B^{-1}I_2^2$ [3]. We denote by Δ_a one of the connected components of the analyticity region of function H_0.

For fixed $I \in \Delta_a$ the perturbing function $H_1(I, \varphi)$ is analytic on the two-dimensional torus $T^2\{\varphi \bmod 2\pi\}$ and is a univalued meromorphic function in

$\mathbf{C} \times \mathbf{C}$ [4, 5]. It is consequently possible to raise in the considered problem the equation of existence of new univalued analytic integrals. Let us consider the case of a nonsymmetric body. i.e. when $A > B > C$.

The integrable unperturbed Euler – Poinsot problem is nondegenerate, since the Hessian $|\partial^2 H_0/\partial I^2|$ is nonzero throughout region Δ_a [3]. The question arises whether contour Γ and the initial conditions $(I^0, \varphi^0) \in \Delta_a \times T^2$, under which the related function $I^1(t; I^0, \varphi^0)$ is not univalued along contour Γ, do exist.

The perturbing function $H_1(I, \varphi)$ may be represented in the form [5]

$$H_1 = f_+(I, \varphi_1) \exp(i\varphi_2) + f_-(I, \varphi_1) \exp(-i\varphi_2) + f_0(I, \varphi_1)$$

For fixed $I \in \Delta_a$ the complex conjugate meromorphic functions $f_+(I, z)$ and $f_-(I, z)$ have a real period of 2π and a pure imaginary "quasi-period" of $i\alpha(I)$ (see [5]). For instance, when the center of mass is located on the major axis of the ellipsoid of inertia, then

$$f_\pm(I, z + i\alpha) = f_\pm(I, z) \exp[\mp \sigma(I)]$$

Explicit expressions for functions $\alpha(I)$ and $\sigma(I)$ are [5]

$$\alpha = \frac{\mathbf{K}'}{\mathbf{K}}, \quad \sigma = \frac{\pi}{\mathbf{K}} F\left(\operatorname{arctg}\frac{\varkappa}{\lambda}, \lambda'\right), \quad \mathbf{K}'(\lambda) = \mathbf{K}(\lambda') \tag{3.2}$$
$$\varkappa^2 = \frac{C(A-B)}{A(B-C)}, \quad \lambda^2 = \varkappa^2 \frac{2CH_0 - I_2^2}{I_2^2 - 2AH_0}, \quad \lambda' = \sqrt{1-\lambda^2}$$

where $\mathbf{K}(\lambda)$ is a complete elliptic integral of the first kind with modulus λ, and F is an elliptic integral of the first kind.

Function $f_0(I, z)$ is elliptic of real and imaginary periods 2π and $i\alpha$, respectively.

Let the ratio of frequencies ω_2/ω_1 be an integral number n when $I = I^0$. If the absolute value $|n|$ is fairly large, such values of "action" variables exist [3], and in that case the real function

$$h(I^0, t) = f_+(I^0, \omega_1 t) \exp(i\omega_2 t) + f_-(I^0, \omega_1 t) \exp(-i\omega_2 t)$$

is periodic of some period T with respect to t. We set

$$h_n(I^\circ) = \frac{1}{T}\int_0^T h(I^\circ, t)\, dt$$

Note that for fixed A, B, and C the mean h_n depends only on n and is independent of I° [3, 6]. Using expansions of functions f_+ and f_- in trigonometric series [5] it is possible to show the existence of an infinite number of nonzero mean quantities h_n (cf. [6]). We denote by $\mathbf{B}_n$ the set of points $I \in \Delta_a$ that satisfy conditions $\omega_2(I) / \omega_1(I) = n$ and $h_n(I) \neq 0$.

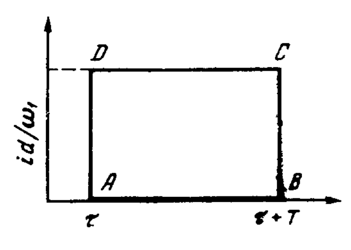

Fig. 1

Let I^0 belong to some $\mathbf{B}_n$ and $\varphi^\circ = 0$. Let us consider in the complex plane $t \in \mathbf{C}$ the closed contour Γ represented in Fig. 1 by the boundary of rectangle ABCD. We select the number τ so that the meromorphic functions

$f_{\pm}(I^\circ, \omega_1 z)$ and $f_0(I^\circ, \omega_1 z)$ do not have poles on Γ. We denote by γ the continuous closed curve in $\mathbf{C} \times \mathbf{C}$ which is the image of the following mapping:

$$\varphi = \omega(I^\circ)t,\ t \in \Gamma \quad (\varphi = (\varphi_1, \varphi_2),\ \omega = (\omega_1, \omega_2))$$

Let U be a small neighborhood of point $I^\circ \in \mathbf{B}_n$ and Ω be a connected region of contour γ in $\mathbf{C} \times \mathbf{C}$, $\Pi(s) \subset \Omega \subset \Pi(S)(0 < s < S)$, such that for all $I \in U$ the meromorphic functions $f_{\pm}(I, \varphi_1)$ and $f_0(I, \varphi_1)$ do not have poles in region Ω.

Theorem 2. There exists for any nonsymmetric solid body an $N(A, B, C)$ such that when point

$$I^\circ \in \mathbf{B} = \bigcup_{|n| \geqslant N} \mathbf{B}_n \subset \Delta_a$$

then the canonical equations of motion of a heavy solid body with a fixed point do not have a univalued integral $F(I, \varphi, \mu)$ that is independent of function (3.1) and analytic in the direct product $U \times \Omega \times (-\varepsilon, \varepsilon)$.

4. **Proof of Theorem 2.** Since the unperturbed problem is nondegenerate, hence by Theorem 1 it is sufficient to establish that function $I^1(t; I^\circ, 0)$ is not univalued. Let us consider the case when the center of mass is located on the major axis of the ellipsoid of inertia. The general case is analyzed similarly.

We set

$$\Phi(I^\circ, t) = -\frac{\partial H_1}{\partial \varphi^2} = -i\,[f_+(I^\circ, \omega_1 t)\exp(i\omega_2 t) - f_-(I^\circ, \omega_1 t)\exp(-i\omega_2 t)]$$

Since

$$I_2^{\boldsymbol{\cdot}} = -\frac{\partial H}{\partial \varphi_2} = -\mu\frac{\partial H_1}{\partial \varphi_2}$$

hence

$$\xi_2 = \oint_\Gamma \Phi(I^\circ, t)\,dt \tag{4.1}$$

Function $\Phi(I^\circ, t)$ is periodic of real period T with respect to t, consequently,

$$\int_B^C \Phi(I^\circ, t)\,dt + \int_D^A \Phi(I^\circ, t)\,dt = 0 \tag{4.2}$$

We set

$$\sigma_{\pm} = \mp i\int_A^B f_{\pm}(I^\circ, \omega_1 t)\exp(\pm i\omega_2 t)\,dt$$

$$\Sigma_{\pm} = \mp i\int_C^D f_{\pm}(I^\circ, \omega_1 t)\exp(\pm i\omega_2 t)\,dt$$

and will show that

$$\Sigma_{\pm} = -\sigma_{\pm}\exp[\mp(n\alpha + \sigma)] \tag{4.3}$$

Substituting variables by formula $t = z + i\alpha / \omega_1$, we obtain

$$\Sigma_{\pm} = \mp \exp[\mp(n\alpha + \sigma)] \int\limits_{\tau+T}^{\tau} f_{\pm}(I^{\circ}, \omega_1 z) \exp(\pm i\omega_2 z)\, dz = - \sigma_{\pm} \exp[\mp(n\alpha + \sigma)]$$

Integral (4.1) with allowance for formulas (4.2) and (4.3) is equal

$$\xi_2 = \sigma_+ [1 - \exp(-n\alpha - \sigma)] + \sigma_- [1 - \exp(n\alpha + \sigma)]$$

Evidently $h_n = i(\sigma_+ - \sigma_-) / T$. Since functions f_+ and f_- are complex conjugate, hence $\bar{\sigma}_+ = \sigma_-$, and since $h_n \neq 0$, the integrals σ_+ and σ_- are nonzero.

Let us show that $\xi_2(I^{\circ}) \neq 0$. Since in the opposite case

$$\left| \frac{1 - \exp(-n\alpha - \sigma)}{1 - \exp(n\alpha + \sigma)} \right| = \left| \frac{\sigma_-}{\sigma_+} \right| = 1$$

consequently $n\alpha(I^{\circ}) + \sigma(I^{\circ}) = 0$. Using formulas (3.2) the above relation can be written as

$$nK'(\lambda) + \pi F\left(\operatorname{arctg} \frac{\varkappa}{\lambda}, \sqrt{1 - \lambda^2}\right) = 0 \tag{4.4}$$

If $|n|$ approaches infinity, $\omega_2 / \omega_1 \to \infty$ and $2H_0 / I_2^2 \to B^{-1}$ [3]. Hence $\lambda \to C / A < 1$ and functions $\mathbf{K}'$ and F tend to definite limits. Since

$$\lim_{\lambda \to C/A} \mathbf{K}'(\lambda) \neq 0$$

and function λ is constant in set $\mathbf{B}_n$ formula (4.4) is invalid when $|n| > N$, $N(A, B, C)$.

Theorem 2 is proved.

It can be shown that a similar statement is also valid when the initial phases φ° are not $(0, 0)$ but $(0, \pi)$. It was shown in [6] that the periodic solutions of the unperturbed problem with initial data $I^{\circ} \in \mathbf{B}_n$, $\varphi_1^{\circ} = 0$, $\varphi_2^{\circ} = 0$, and π do not vanish with the addition of perturbation and for small values of parameter $\mu \neq 0$ become the nondegenerate periodic solutions of the perturbed system of equations. Thus the calculations carried out in Section 4 prove that beginning from some number n the nondegenerate periodic solutions for small $\mu \neq 0$, determined in [6] are not univalued functions in the complex time plane.

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